Knots, links, spatial graphs and their rep var.

Y 3-mfd

$\pi_1(Y) \to SU(2),\ SO(3),\ SL_2\mathbb{C}$

Casson invariant

closed Y $H_1(Y,\mathbb{Z})=0$

$$\lambda(Y) := \frac{1}{2}\#\{\text{irreducible } \pi_1(Y)\to SU(2)\}/_{\text{conjugate}}$$

Casson-Lin invariant

knot $K \subseteq S^3$

$$\lambda(K) := \#\underline{\{\text{irr } \pi_1(S^3-K)\to SU(2)} \mid \text{tr } \rho(\text{meridian}) = 0\}$$

$$\lambda(K) = \frac{1}{2}\,\text{signature}(K)$$

$I^{\natural}$

Question

$\pi_1(Y)$ non-abelian $\exists$ irr $\rho: \pi_1(Y) \longrightarrow SU(2)$?

Thm(Kronheimer - Mrowka , 2010)

$K \subseteq S^3$ non-trivial , then

$\exists$ meridian-traceless irr rep

$\rho: \pi_1(S^3 - K) \to SU(2)$.

$R(K) := \{ \rho: \pi_1(S^3 - K) \to SU(2) \mid \underline{tr\, \rho(\text{meridian}) = 0} \} / \text{conjugate}$

$R^{\#}(K) := \{ \rho: \pi_1(S^3 - K) \to SU(2) \mid tr\, \rho(m) = 0 \}$

$\#$ sharp $\qquad \rho(m) \sim i \in SU(2)$

$\underline{R^{\natural}(K) := \{ \rho: \pi_1(S^3 - K) \to SU(2) \mid \rho(m) = i \}}$

$\natural$ natural

Instanton Floer homology $\qquad I^{\natural} \quad \mathbb{Z} \times \mathbb{Z}/2$

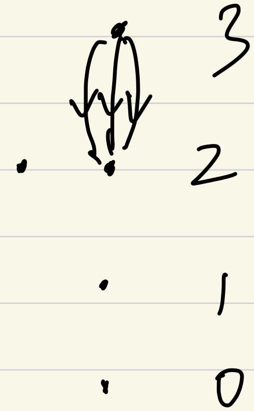
$$I^{\natural}(K) = H_*(\mathbb{Z}R^{\natural}(K), d)$$

$$\chi(I^{\natural}(K)) = \text{Alexander}(K) = \Delta_K(t)$$

$$f : R^{\natural}(K) \to \mathbb{R} \quad \text{Morse}$$

$$\underline{I^{\natural}(K) = H_*(\mathbb{Z}\,\text{Critical}(f), d)} \xLeftarrow{S.S.} H_*(R^{\natural}(K))$$

$$M \xrightarrow{f} \mathbb{R}$$



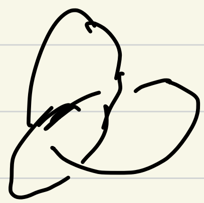
$$CS : B(S^3 - K) \to S^1$$

Chern-Simons

Thm $\underline{\dim I^{\natural}(K) = 1} \iff K = \text{unknot}$

$$\text{Jones} \xLeftarrow{\chi} \text{Khovanov}(K) \Longrightarrow I^{\#}$$

$$\text{Khr}(K) \Longrightarrow I^{\natural}$$

T  $\pi_1 = \langle x_1, x_2, x_3 \mid x_2 x_1 x_2^{-1} = x_3 \}$

$$R(T) = \langle \underbrace{\theta}_{\text{abelian}}, \underbrace{\rho}_{\text{irr}} \rangle$$

$$R^{\natural}(T) = * \sqcup S^1$$

Thm (Baldwin-Sivek, 2018)

$R(K) = \{\theta, \rho\} \Rightarrow K = \text{trefoil}$

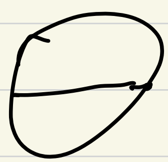
ρ non-degenerate

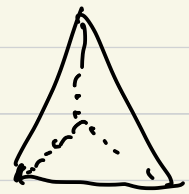
Thm (X-Zhang, 2021)

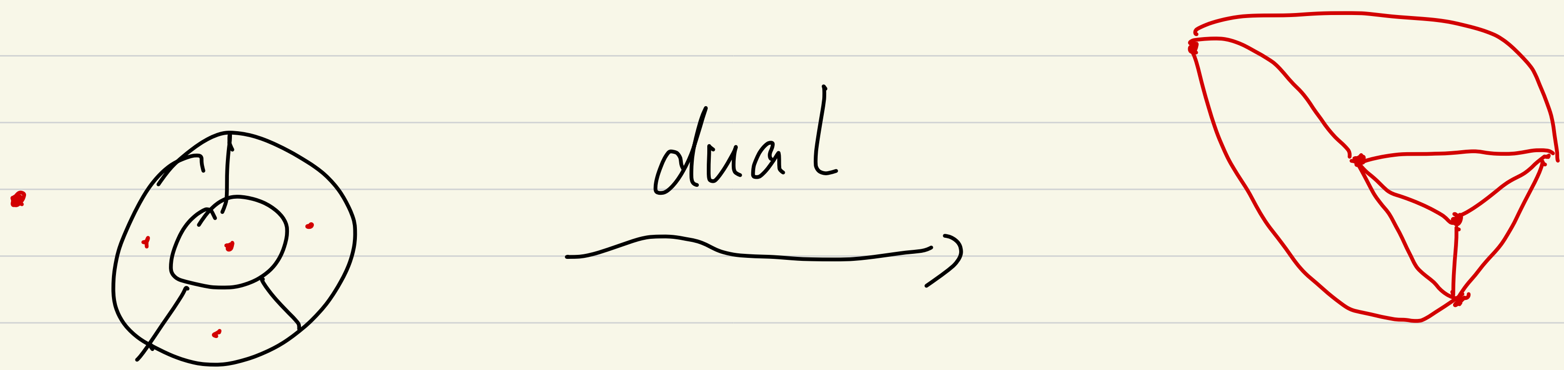
If L is not Hopf link, then

$\exists$ irr $\rho : \pi_1(S^3 - L) \to SU(2)$.

Web: spatial cubic/trivalent graph







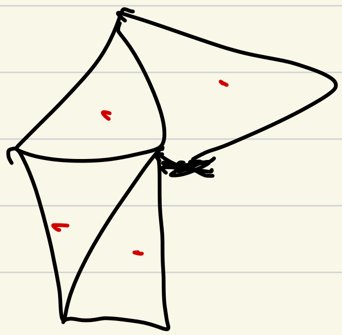
G_0 planar graph 4-face-colorable (4 color thm)

$\Longleftrightarrow$ $G_1 = \mathrm{dual}(G_0)$ 4-vertex-colorable

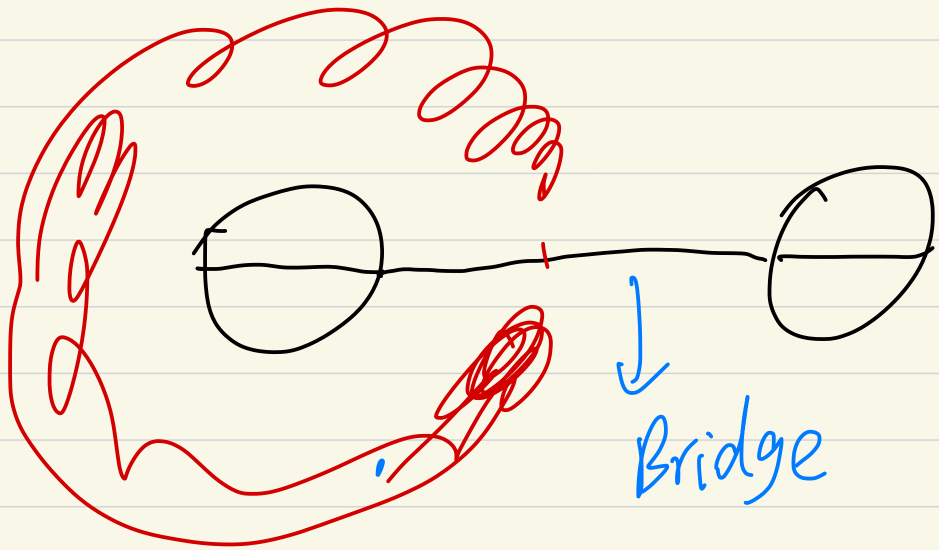
$\Longleftarrow$ G_2 = triangulation of G_1 4-vertex-colorable

$\Longleftrightarrow$ $G_3 = \mathrm{dual}(G_2)$ 4-face-colorable

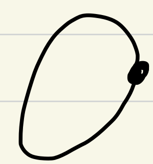
G_2 dual, G_3



cubic graph
bridgeless



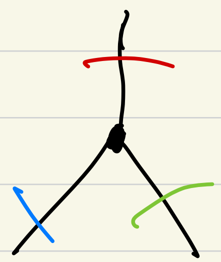
dual



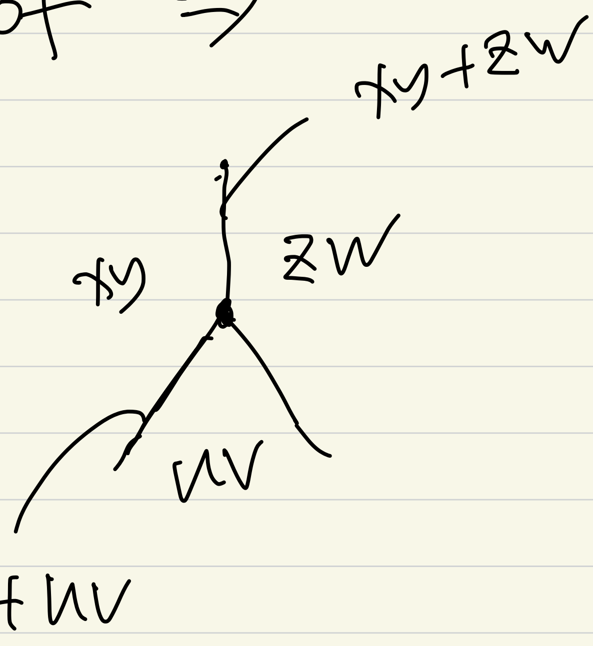
Thm (Tait)

G_3 4-face-colorable $\Longleftrightarrow$ G_3 3-edge-colorable

(Tait coloring)



Proof $\Rightarrow$



A	B	C	D
00	01	10	11

XOR operation

$xy \wedge zw = \overline{(x+z)(y+w)}$

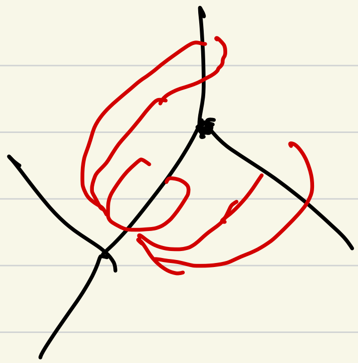
$01 \wedge 10 = 11$

$11 \wedge 10 = 01$

$\square$

Γ web

$$R^{\#}(\Gamma) := \left\{ \rho : \pi_1(S^3-\Gamma) \to SO(3) \;\middle|\; \rho(m_i) \sim \begin{pmatrix}-1&&\\&-1&\\&&1\end{pmatrix} \ \forall i \right\}$$



$H = \operatorname{Im}\rho$

$$O(2) \subseteq SO(3)$$
$$O(2) = \left\{ \begin{pmatrix} A & \\ & \pm 1 \end{pmatrix} \right\}$$

$Z(H)$	H
$O(2)$	$\mathbb{Z}/2 = \left\langle \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}, \begin{pmatrix}-1&&\\&-1&\\&&1\end{pmatrix} \right\rangle$
$\left\{ \begin{pmatrix}-1&&\\&-1&\\&&1\end{pmatrix}, \begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix}, \begin{pmatrix}-1&&\\&1&\\&&-1\end{pmatrix}, \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \right\} = V_4$	V_4
$\mathbb{Z}/2$	$V_4 < H \subseteq O(2)$
$\{1\}$	$gHg^{-1} \not\subseteq O(2)$ fully irreducible

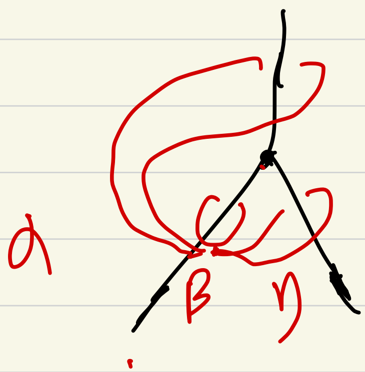
oberservation (Kronheimer-Mrowka)

$$\{ \rho \in R^{\#}(\Gamma) \mid \operatorname{Im}\rho \subseteq V_4 \}$$

$$\Big\updownarrow \text{1-1}$$

$$\{ \text{Tait coloring of } \Gamma \}$$

Proof



$\beta\gamma = \alpha$

$\beta \mapsto$: $V_4 - \{1\}$

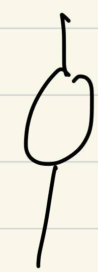
$\gamma \mapsto$

$\alpha \mapsto$

Lemma (Kronheimer - Mrowka)

$R^\#(\Gamma)$ Γ planar contains a $O(2)$-rep

$\Rightarrow$ $\exists$ V_4-rep



$$H_*(R^\#(\Gamma)) \overset{s.s.}{\Longrightarrow} J^\#(\Gamma)$$

Conjecture $\dim J^\#(\Gamma) = \#\mathrm{Tait}(\Gamma)$ Γ planar

Thm (Kronheimer - Mrowka)

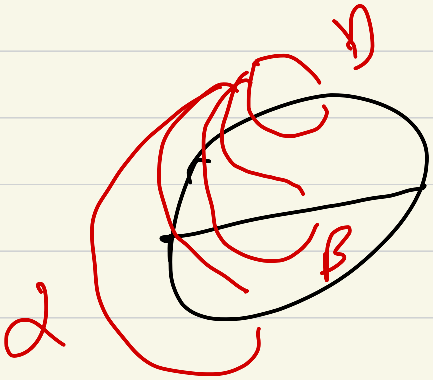
Γ web no embedded bridge

$\Rightarrow$ $J^\#(\Gamma) \neq 0 \Rightarrow R^\#(\Gamma) \neq \phi$.

Γ planar

Thm $\dim J^{\#}(\Gamma) \geq \# \mathrm{Tait}(\Gamma)$

$R(\Theta) = R^{\#}(\Theta)/\text{conjugate} = \{*\}$



$$\alpha\beta = \gamma$$

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & & \\ & +1 & \\ & & -1 \end{pmatrix}$$

Thm(X) Γ spatial theta graph

$R(\Gamma) =$ single point (non-degenerate)

$\Rightarrow$ Γ planar theta graph

$\varphi : \mathbb{R} \to \mathcal{B}(Y)$

$$\frac{d\varphi}{dt} = \operatorname{grad} CS_{\varphi}$$

$$\Updownarrow$$

$\mathbb{R} \times Y$ Anti-self-dual YM equation