

Cluster Nature of Quantum Groups

arXiv: 2209.06258

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October 27, 2022

1. Basic setup

- G : complex group of type ADE
with trivial center
(e.g. $G = \mathrm{PGL}_n(\mathbb{C})$)
- $\mathfrak{g} = \mathrm{Lie} G$
- $U_q(\mathfrak{g})$ quantum group / quantized universal
enveloping algebra of \mathfrak{g} .

Goals

- a) present a rigid cluster model realizing $U_q(\mathfrak{g})$
- b) Construct a "cluster canonical basis" for $U_q(\mathfrak{g})$

2. Moduli space of G -local systems



S : oriented surface with
punctures inside
& marked points on its boundary

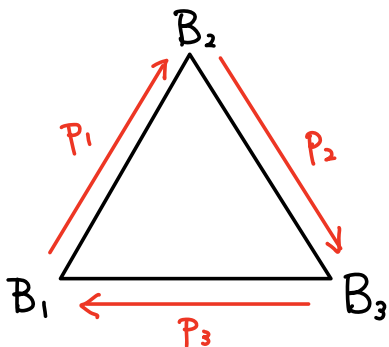
[Goncharov - S, 19] introduced the moduli space

$$\mathcal{P}_{G,S} = \left\{ \begin{array}{l} G\text{-local systems } \mathcal{L} \text{ over } S \\ + \text{ a choice of a flat section} \\ \text{on } \mathcal{L} \times_G (G/B) \text{ for each} \\ \text{marked point \& puncture} \\ + \text{ a choice of a pinning} \\ \text{for every boundary interval} \end{array} \right\} / G$$

Remark. $\mathcal{P}_{G,S}$ is an enhancement of the celebrated Fock-Goncharov moduli space $\mathcal{X}_{G,S}$.

Examples

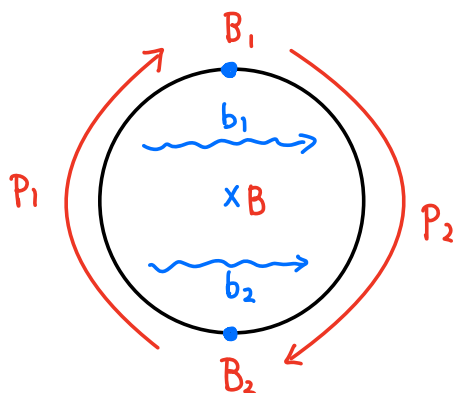
a) $t =$ a triangle / a disk with three marked pts



$\mathcal{P}_{G,t}$ consists of the left data, where B_1, B_2, B_3 are Borel subgps of G such that $B_i \cap B_j$ is a commutative subgroup, $\forall i \neq j$

The pinning P_i is a choice of a Chevalley basis of $\mathfrak{g} = \text{Lie } G$ w.r.t. (B_i, B_{i+1})

b) $S = \odot$ a once punctured disk with two marked pts



Fix a pair of opposite Borel subgroups B^+, B^- of G

Let $\mathcal{B} \cong G/B^+$ be the flag variety consisting of all Borel subgroups of G

Then

$$\mathcal{P}_{G,0} \cong \left\{ (b_1, b_2, B) \in B^+ \times B^- \times \mathcal{B} \mid b_2^{-1} b_1 \in B \right\}$$

There is a natural projection

$$\mathcal{P}_{G,0} \longrightarrow B^+ \times B^-$$

whose generic fibers are Weyl group torsors

Remark. One may compare it with the usual Grothendieck - Springer resolution

$$\begin{array}{ccc}
 \widehat{G} & (g, B) & \curvearrowright W \text{ acts by alternating } B. \\
 \downarrow & \downarrow & \\
 G & \mathfrak{g} & \\
 \Rightarrow & \mathcal{O}(\widehat{G})^W = \mathcal{O}(G) &
 \end{array}$$

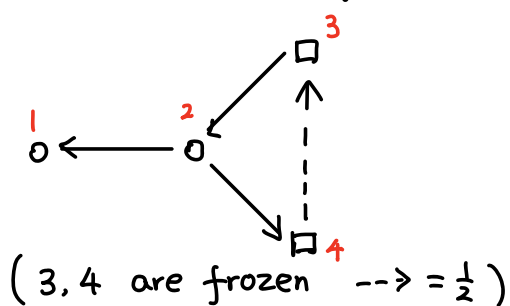
Theorem ([Goncharov - S, 19], [S20])

The ring $\mathcal{O}(P_{G,S})$ of regular functions of $P_{G,S}$ is a cluster Poisson algebra, and therefore it admits a quantization $\mathcal{O}_q(P_{G,S})$

3. Fock - Goncharov quantum cluster algebra

- seed

Q : an ice quiver



exchange matrix

$$E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\mathbb{L} := \mathbb{Z}[q^{\pm \frac{1}{2}}, q^{-\frac{1}{2}}]$$

T_Q : an \mathbb{L} -algebra generated by $X_i^{\pm 1}$,
where i are vertices of Q .

with relations

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i$$

$\Sigma_Q := (Q, T_Q, \{X_i\})$ is called a seed.

- Mutations.

Each mutable k gives a new $\{X'_i\}$

such that

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i \prod_{r=1}^{|\varepsilon_{ik}|} (1 + q^{2r-1} X_k) & \text{if } \varepsilon_{ik} < 0, k \neq i \\ X_i \prod_{r=1}^{|\varepsilon_{ik}|} (1 + q^{2r-1} X_k^{-1})^{-1} & \text{if } \varepsilon_{ik} \geq 0, k \neq i \end{cases}$$

→ quantum torus algebra $T_{Q'}$ generated
by $X_i^{\pm 1}$ with relations

$$X_i' X_j' = q^{2\varepsilon_{ij}'} X_j' X_i'$$

where (ε_{ij}') is the exchange matrix for
 $Q' = \mu_k(Q)$

→ mutated seed

$$\Sigma_{\mu_k(Q)} = (Q', T_{Q'}, \{X_i'\})$$

In general, we say $\Sigma_{Q'} \sim \Sigma_Q$ if $\Sigma_{Q'}$

can be obtained from Σ_Q by a sequence
of mutations

- Quantum cluster algebra is a $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -algebra

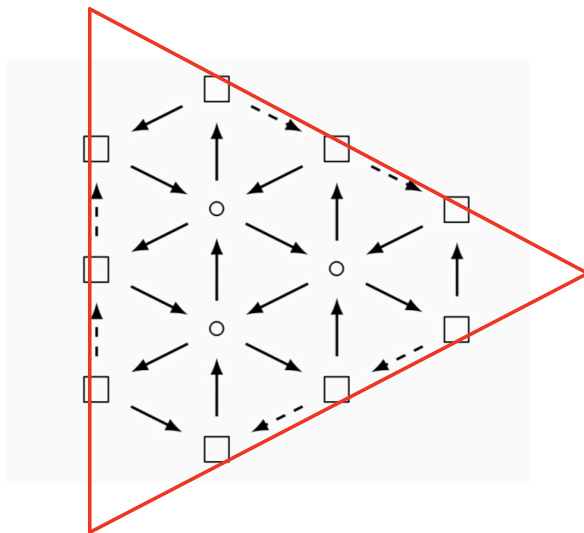
$$\mathcal{O}_q(\Sigma_Q) := \bigcap_{\Sigma_{Q'} \sim \Sigma_Q} T_{Q'}$$

(The semiclassical limit of $\mathcal{O}_q(\Sigma_Q)$ gives the usual cluster Poisson Alg)

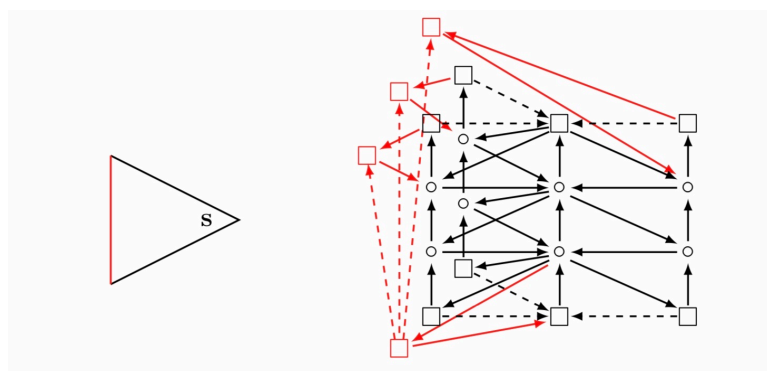
Thm (Davison - Mandel) If the mutable part of Q admits a reddening sequence, then $\mathcal{O}_q(\Sigma_Q)$ admits a natural basis, equivariant under the action of its cluster modular group.

Example Cluster structure on $\mathcal{P}_{G,t}$

$G = \mathrm{PGL}_4$



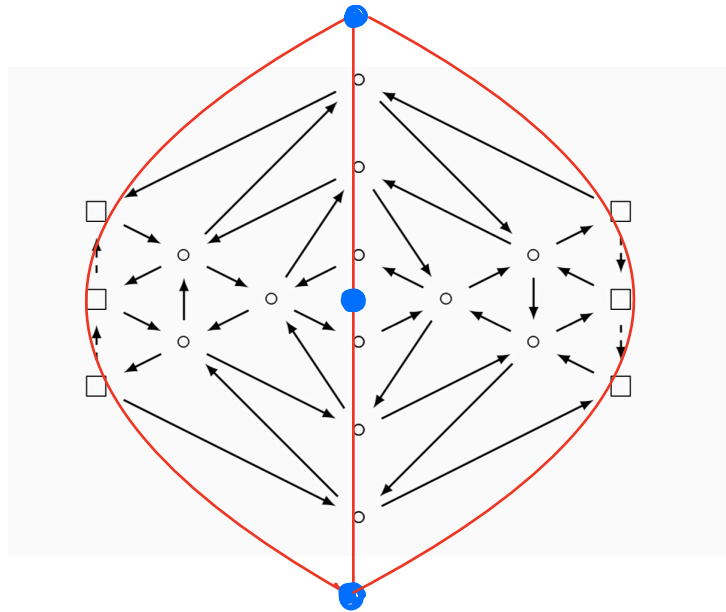
$G = \mathrm{PSO}_8$
(type D_4)



The quiver $Q_{G,t}$ for $\mathcal{P}_{G,t}$ has $3r$ many frozen vertices, where $r = \text{rk } G$

(The black part coincides with the quiver for the double Bruhat cell $G^{w_0, e}$)

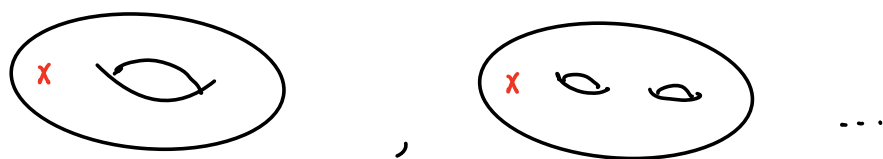
$\mathcal{P}_{\text{PGL}_4, \odot}$



For general $\mathcal{P}_{G,S}$, we shall consider an ideal triangulation of S . We place a quiver $Q_{G,t}$ on every triangle and then glue them along corresponding edges.

Theorem (Goncharov-S, 14, 18, 19)

- 1) $\mathcal{O}(\mathcal{P}_{G,S})$ is a cluster Poisson algebra
- 2) The following groups act on $\mathcal{O}(\mathcal{P}_{G,S})$ via (quasi)-cluster automorphisms
 - the mapping class group of S
 - the outer automorphism group $\text{Out}(G)$ of G
 - the product of Weyl groups over punctures
 - the product of Braid groups Br_G over boundary circles of S
- 3) Suppose S is not a once punctured surface with no boundaries



Then the quiver for $\mathcal{P}_{G,S}$ has a reddening sequence

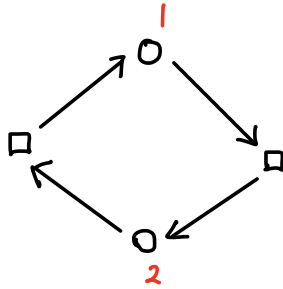
(Remark. Fraser-Pylyavskyy: For $G = \mathrm{PGL}_n, n \geq 3$,
3) also holds for once punctured closed
surfaces)

4. Cluster model for quantum group

- $S = \textcircled{\bullet}$ $\mathcal{O}(\mathcal{P}_{G,0})$ is a cluster Poisson algebra
 ↓ quantization
 $\mathcal{O}_q(\mathcal{P}_{G,0})$ quantum cluster algebra

- Recall the Weyl group action on $\mathcal{P}_{G,0}$
Thm ([GS18]) W acts on $\mathcal{O}(\mathcal{P}_{G,0})$ via
cluster automorphisms.

Example. $G = \mathrm{PGL}_2$



$W = \mathbb{Z}/2$ acts via $\pi_{12} \circ \mu_2 \circ \mu_1$, where π_{12} switches 1 & 2.

We can lift the action of W to $\mathcal{O}_q(\mathcal{P}_{G,0})$ and define the W -invariants

$$\mathcal{O}_q(\mathcal{P}_{G,0})^W$$

- Recall the projection

$$\begin{aligned} \mathcal{P}_{G,0} &\longrightarrow \mathcal{B}^+ \times \mathcal{B}^- \longrightarrow H \times H \longrightarrow H \\ (b^+, b^-, B) &\longmapsto (b^+, b^-) \longmapsto (h^+, h^-) \longmapsto h^+ h^- \\ &\quad \quad \quad \parallel \quad \parallel \\ &\quad \quad \quad u^+ h^+ \quad u^- h^- \end{aligned}$$

It gives rise to regular functions

$$O_i := \alpha_i(h^+ h^-) \quad i=1, \dots, r$$

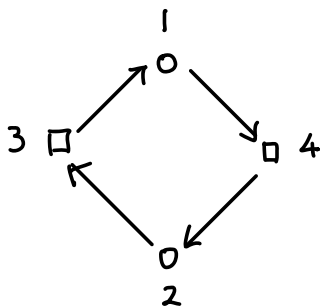
} simple positive root

Lemma. O_1, \dots, O_r are Casimir elements on $\mathcal{O}(\mathcal{P}_{G,0})$
and can be lifted to central elements

$$\mathbb{O}_1, \dots, \mathbb{O}_r \in \mathcal{O}_q(\mathcal{P}_{G,0})^W$$

Let $I \subset \mathcal{O}_q(\mathcal{P}_{G,0})^W$ be the ideal generated
by $\mathbb{O}_1 - 1, \mathbb{O}_2 - 1, \dots, \mathbb{O}_r - 1$.

Example $G = \mathrm{PGL}_2$



$$O = \chi_1 \chi_2 \chi_3^2 \chi_4^2$$

Theorem ([S22])

1. We have a natural Hopf algebra isomorphism

$$U_q(\mathfrak{g}) \cong \mathcal{O}_q(\mathcal{P}_{G,0})^w / I$$

2. $U_q(\mathfrak{g})$ admits a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ -linear basis $\overline{\mathcal{H}}$ such that

- The structural coefficients of $\overline{\mathcal{H}}$ are in $\mathbb{N}[q^{\pm\frac{1}{2}}]$
- $\overline{\mathcal{H}}$ is equivariant under Lusztig's braid group action on $U_q(\mathfrak{g})$ & the action of $\text{Out}(G)$ (\in Dynkin automorphism)

• Every element in $\overline{\mathcal{H}}$ is self-adjoint

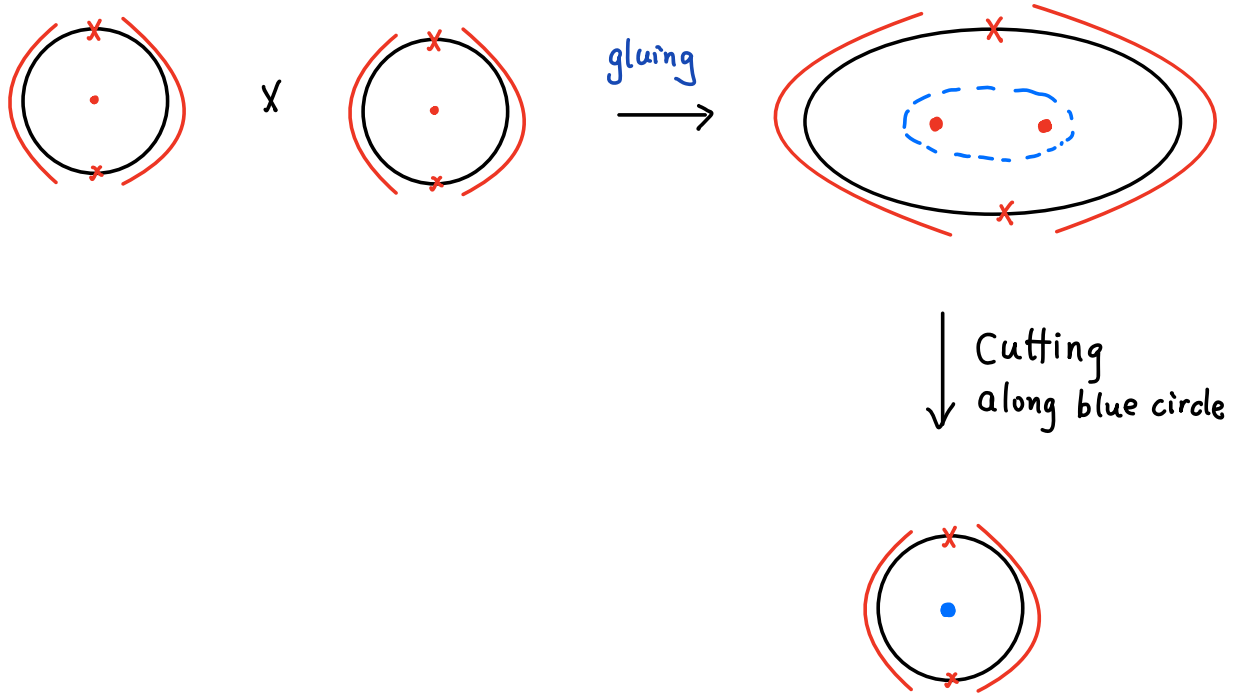
• The basis $\overline{\mathcal{H}}$ is naturally parametrized by the set $U_+^L(\mathbb{N}) \times X^*(H) \times U_+^L(\mathbb{N})$

where $U_+^L(\mathbb{N})$: Lusztig data for the langlands dual group

$X^*(H)$: weight lattice of G

Remark

- Coproduct structure



$$\Rightarrow \Delta: \mathcal{O}(P_{G,0})^w \rightarrow \mathcal{O}(P_{G,0})^w \otimes \mathcal{O}(P_{G,0})^w$$

Its quantized version coincides with the
coproduct structure on $U_q(\mathfrak{g})$

✱

Thank You!