Kapustin-Witten Eqs and higher Teichmülle theory

(joint with R. Mazzeo).

I KW eqs

II Witten's program.

III ~~Ex~~Extended Bogomolny eqs (EBE).

IV. EBE & higher Teichmüller theory

I KW eqs (06)

$$\begin{array}{c} P \leftarrow G \\ \downarrow \\ M \end{array}$$

M closed 4-mfld $\quad G = SU(n)$

$P = G \times M$.

A connection $\quad \phi$ $\mathfrak{g}$-valued 1-form

Φ 1-form $\mathfrak{g}$-valued

$$\begin{cases} F_A - \Phi \wedge \Phi + * d_A \Phi = 0 \\ d_A * \Phi = 0 \end{cases}$$

Hodge stare

$\Lambda^2 \to \Lambda^2$.

F_A Curvature $\quad dA + A \wedge A$

- Elliptic $\quad \mathrm{Ind} = -\dim(\mathfrak{g})\,\chi(M)$

- $SL_n(\mathbb{C})$ gauge theory equation

$$\underline{sl_n(\mathbb{C})} = \underline{su(n)} \oplus i\,\underline{su(n)}$$

$$\mathcal{A} = A + i\Phi$$

- YM eqs. $\quad M = \mathbb{R}_t \times Y \quad Y$ 3-mfld.

$$\frac{\partial}{\partial t} A = \nabla CS(A)$$

KW eqs $\quad M = \mathbb{R}_t \times Y$

$$\frac{d}{dt}\mathcal{A} = \nabla \operatorname{Im}(CS^{\mathbb{C}}(\mathcal{A}))$$

- $M = \mathbb{R}^4 \qquad (\mathbb{R}^4 \otimes \mathbb{C}) \simeq \mathbb{C}^4 \cong T^*\mathbb{R}^4$

$T^*\mathbb{R}^4$ YMeq, $\mathbb{R}^4$-direction inv.

$\Rightarrow$ KW eqs.

<u>Thm</u> (KW 06).

M closed 4-mfld, then all sols

~~to~~ KW eq

$$\begin{cases} F_A - \Phi\wedge\Phi + * d_A\Phi = 0 \\ d_A * \Phi = 0 \end{cases} \longleftrightarrow \begin{cases} F_A - \Phi\wedge\Phi = 0 \\ d_A\Phi = 0 \\ d_A * \Phi = 0 \end{cases} \Leftrightarrow F_{\mathcal{A}} = 0$$

~~A~~ flat $SL_n(\mathbb{C})$ connection

$\leftrightarrow \{ \pi_1(M) \to SL_n(\mathbb{C}) \} / \sim$

II Witten's Program 09'.

$M = Y \times \mathbb{R}^+_t = (0,\infty)$ t-coordinate

$K \subset Y \times \{0\}$

- counting sols with $\boxed{\text{bdry condition}}$

$\Rightarrow$ Jones polynomial.

- $K = \emptyset$. Invariant on 3-mfld.

- Haydys-Witten over $Y \times \mathbb{R} \times \mathbb{R}^+$

Khovanov-homology.

(Jones conj). $J(K) = 1 \overset{?}{\Rightarrow} K = \bigcirc$?

Nahm pole bdry condition.

$(y,t) \in Y \times \mathbb{R}^+$

$t \to 0$ $\quad y \to K$ $\quad \Phi \sim \frac{e}{t} + \cdots$ $\quad \frac{1}{t}$

$e : TY \to \mathfrak{g} \subset \overset{n}{\mathfrak{su}(2)}$ isom

$dy_1, dy_2, dy_3 \qquad [t_i, t_j] = \frac{1}{2}\epsilon_{ijk} t_k$

$e = t_1 dy_1 + t_2 dy_2 + t_3 dy_3$

$t \to 0$ $\quad y \to K$ (A, Φ) like a

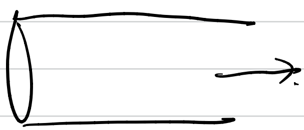
model solution distinguish

~~the~~ knot positition $\Phi \sim \frac{1}{\mathrm{dist}(K)}$

$K = \emptyset$, $Y = \mathbb{R}^3$ $\left(A = 0, \Phi = \frac{e}{t}\right)$

is a solution to KW (Nahm 1978)

$Y \times \mathbb{R}^+$ $\qquad t \to 0$ $\quad \Phi \sim \infty$ $\quad$ Nahm pole.



$t \to \infty$ at cylinderical end.

$(A, \Phi) \longrightarrow$ flat $SL_n(\mathbb{C})$ connection ρ

$$M_\rho = \left\{ \text{sols} \;\middle|\; \begin{array}{l} t \to 0 \ \text{NP bdry cond} \\ t \to \infty \ \text{convergence to } \rho \end{array} \right\}$$

$$k = \frac{1}{4\pi^2} \int_{Y \times \mathbb{R}^+} \mathrm{Tr}(F_A \wedge F_A)$$

$$\mathcal{M}_\rho = \coprod_k \mathcal{M}_{\rho,k}$$

(Witten Conj)

counting

$$J_\rho(K) = \sum_k \#(\mathcal{M}_{\rho,k})\, q^k$$

$$\dim \mathcal{M}_{\rho,k} = \chi(Y \times \mathbb{R}^+) = 0$$

$S^3 \quad \pi_1(S^3) \to SL_n(\mathbb{C}) \quad J_{\rho_0}(K)$

$Y^3 \quad \pi_1(Y^3) \to SL_n(\mathbb{C}) \quad J_\rho(K)$

- ρ & flat $SL_n(\mathbb{C})$ conn

  $[\rho]$ is generator of $SL_n(\mathbb{C})$ Floer homology

  (Manolesu-Abouzaid) Casson inv homology

- $J(K) = \sum J_\rho(K)\, [\rho]$

  polynomial with coefficients of generators of $SL_n(\mathbb{C})$ Floer homology. $(n=2)$

$$J(\bigcirc) = q^{1/2} + q^{-1/2}$$

Difficulty ① singular bdry condition

$$\mathcal{L} = D_1 + \frac{1}{t} D_0$$

operator

D_1 linearization 1st elliptic

D_0 good 0-order operator

bad coefficients

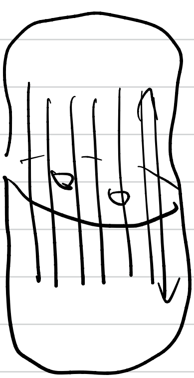
② Non-cptness

$$\#(\mathcal{M}_{e,k})$$

Taubes use $\mathbb{Z}_2$-harm 1-forms

singularity.

III Gaiotto-Witten's approach (2011) $SO(2)$



Y $K = \phi$

$E \leftarrow \underline{\mathbb{C}}^2$, H metric

$\downarrow$ $\det E = 0$

$\Sigma_g \rightarrow$ KW eqs over $\Sigma_g \times \mathbb{R}^+_t$

Extended Bogomolny eqs.

A conn ϕ 1-form $\mathfrak{g}$ & ξ 0-form $\mathfrak{g}$

$$\begin{cases} F_A - \phi\wedge\phi = * d_A \xi \\ d_A\phi + *[\phi, \xi] = 0 \\ d_A^{*}\phi = 0 \end{cases}$$

$\xi = 0$ Hitchin eq.

$\phi = 0$ Bogomolny eq. $F_A = * d_A \xi$

Σ-inv Nahm eq.

$D_1 = \bar{\partial}_\Sigma \quad D_2 = \phi^{1,0} \quad D_3 = \nabla_t - i\xi$

$$\text{EBE} \Longleftrightarrow \begin{cases} [D_i, D_j] = 0 \\ \sum_{i=1}^{3} [D_i, D_i^{\dagger}] = 0 \end{cases}$$

adjoint.

$g \in Aut(E) \quad GL(2;\mathbb{C})$.

$D_i \longrightarrow g^{-1}\circ D_i \circ g$ action

preserve 1st eqs

Moment map eq.

$\mathbb{C}^2$ Köhler mfld $z_1, \dots, z_n$

$$D_i = \bar{\partial}_{z_i}$$

$$\begin{cases} [D_i, D_j] = 0 \\ \sum_{i=1}^{n} [D_i, D_i^\dagger] = 0 \end{cases} \qquad \begin{cases} F_A^{0,2} = 0 \\ \Lambda F_A^{1,1} = 0 \end{cases}$$

HYM (DUY thm)

* Geom meaning of the 1st equation

$D_1 = \bar{\partial}_\Sigma \quad D_2 = [\varphi, \cdot\,] \quad \varphi = \phi^{1,0}, \; D_3 = \nabla_t - i\xi.$

$[D_1, D_2] = 0 \Leftrightarrow \bar{\partial}_\Sigma \varphi = 0 \quad (\bar{\partial}_\Sigma, \varphi)$

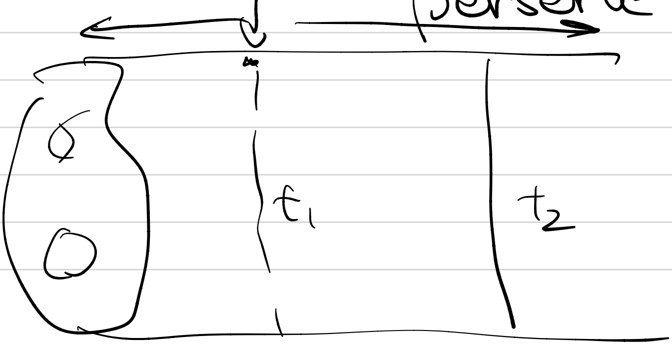
Higgs bundle

D_3 parallel transport on t-direction

$[D_3, D_1] = 0 \; [D_3, D_2] = 0 \Leftrightarrow$ parallel transport

$(\mathcal{E}, \varphi)$

perserve Higgs bundle



bdry condition ~~$SL_2(\mathbb{C})$~~ . $\mathbb{C}^2$

$$\varphi = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz .$$

$$D_t = \partial_t + \frac{1}{2t}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad t \to 0$$

$D_t s = 0$ s section $s = \begin{pmatrix} a t^{-1/2} \\ b t^{1/2} \end{pmatrix}$

a, b constant

$t \to 0$ $a = 0$, $b \neq 0$

exists a vanishing line bundle

L $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\mathrm{Span}\{L_z^{\otimes K}, \varphi(L)\} = E \otimes K$ at all points

$$L \xrightarrow{\varphi} E \otimes K \to \underbrace{(E/L)}_{\cong L^{-1}} \otimes K$$

$$L \cong L^{-1} \otimes K \Rightarrow L \cong K^{1/2}$$

$$\mathcal{E} = K^{-1/2} \oplus K^{1/2}$$

$$\varphi = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix} \quad \beta : K^{-\frac{1}{2}} \to K^{\otimes 1/2} \otimes K$$

$\beta \in H^0(K^2)$

$1: K^{1/2} \to K^{-1/2} \otimes K = K^{1/2}$

Hitchin component.

$\rho : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \quad e = 2g-2\,?$

$\mathbb{C}^n$ from NP $\Rightarrow$ Higher Teichmüller component

Conj

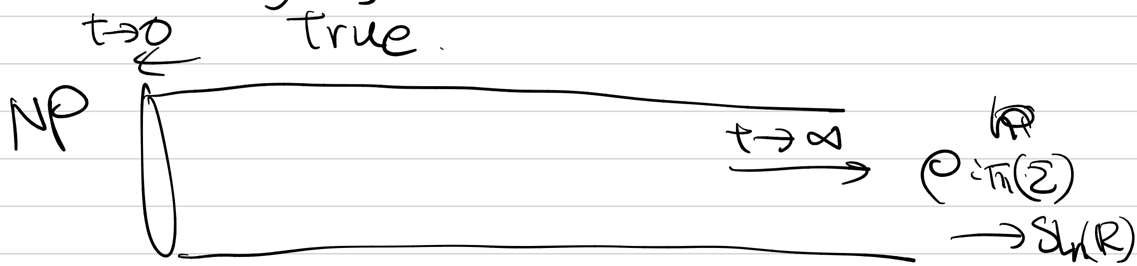
{Gaiotto-Witten}

{Sols to EBE with Nahm Pole $SU(n)$}$/\mathcal{G}$

$\longleftrightarrow$ {Teichmüller component of $PSL_n(\mathbb{C})$}$/\sim$

Thm (H.-Mazzeo)

The Conj of Gaiotto and Witten is true.



① Find a metric H $\quad \Omega_H = 0$

Step 1: Find good approximate solution H_0

$$H = H_0 e^{s}$$

Step 2: C^0-estimate

$$\Delta |s|^2 \leq M |\Omega_{H_0}| \quad M>0$$

$$\Rightarrow |s|_{C^0} \leq C$$

Step 3: $\Omega_H = 0$ by method of continuation

$$\left(\Delta + \frac{1}{t^2}\right) s + B\left(\left(\nabla + \frac{1}{t}\right) s\right) + C\left(\left(\nabla + \frac{1}{t}\right) s \otimes \left(\nabla + \frac{1}{t}\right) s\right) = 0$$

$$(t^2 \Delta + 1) s + B((t\nabla + 1) s) + C((t\nabla + 1) s \otimes (t\nabla + 1) s) = 0$$

rescale invariant

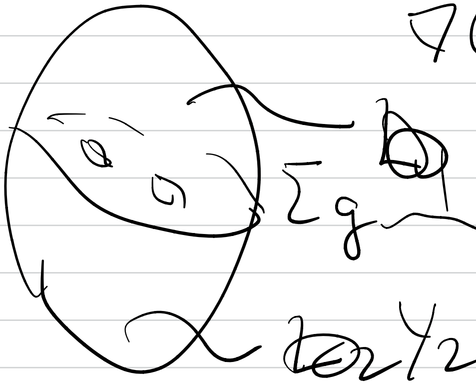
& higher derivative estimate

find a solution.

SU(2) Floer theory

Atiyah-Floer conjecture

∇CS Morse theory



KW

$\{\pi_1(\Sigma_g) \to SU(2)\}$ {Tec in}

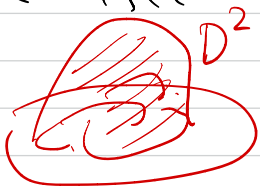
$L_i = \{\pi_1(Y_i) \to SU(2)\}$

{Floer homology}

$) \; L_1 \cap L_2 \} \cong$ ~~Morse ther~~

Fukaya Ali.

T^*S^3



$\cup$ - Ekhlom

$K := \Phi \quad J(a) \qquad ? \qquad q = e^{i\frac{\pi}{k}}$

$\rho : \pi_1(Y) \to PSL_2(\mathbb{C})$