Bounded differentials on unit disk and the associated geometry

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- Let S be a closed surface with genus at least 2.
- Let g, h be two hyperbolic metrics on S.
- Let $f : (S, g) \to (S, h)$ be a harmonic map.
- Then $q = (f^*h)^{(2,0)}$ is a holomorphic quadratic differential on (S, g), called the Hopf differential.

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Conversely,

Theorem (Hitchin 87', Wolf 89')

Given a holomorphic quadratic differential q on (S, g), there is a unique harmonic map $f : (S, g) \to (S, h)$ such that q is the Hopf differential of f.

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  (1) geometric object (harmonic map),
  (2) holomorphic object (quadratic differential).

- The theorem above relates two kinds of objects:
  (1) geometric object (harmonic map),
  (2) holomorphic object (quadratic differential).
- We further study the boundedness relation of these objects.
- For example, if $|q|_g \leq C_1$, is there a constant C_2 depending on C_1 such that the energy density $|df|^2 \leq C_2$?

Lifting to the universal cover, we consider the Poincaré disk $\mathbb{D}$.

Theorem (Wan 92')

Let $f : \mathbb{D} \to \mathbb{D}$ be a harmonic map with the Hopf differential q. Suppose $\hat{g} = |\partial f|^2 g_{\mathbb{D}}$ gives a complete metric. Then the followings are equivalent.

(1) The differential q is bounded with respect to $g_{\mathbb{D}}$.

(2) The energy density $|df|^2$ is bounded.

(3) The harmonic map f is quasi-conformal.

Remark

For "bounded" here, we mean not only the boundedness but also the estimates.

In this talk, we generalize Wan's result to more holomorphic differentials and geometries.

- Cubic differential q_3: hyperbolic affine spheres in $\mathbb{R}^3$. (Loftin 01', Labourie 07')
- Quartic differential q_4: maximal surfaces in $\mathbb{H}^{2,n}$. (Collier-Tholozan-Toulisse 19')
- Sextic differential q_6: J-holomorphic curves in $\mathbb{H}^{4,2}$. (Baraglia 10', Evans 22', Nie 22')

Theorem (D-Li 22')

In the three geometries above, let the surface X (hyperbolic affine sphere, maximal surface, J-holomorphic curve) be the image of a conformal immersion of $\mathbb{D}$. Suppose the induced metric g is complete. Then the followings are equivalent.
(1) The differential $q = q_3, q_4, q_6$ is bounded with respect to $g_{\mathbb{D}}$.
(2) The induced metric g is bounded by $g_{\mathbb{D}}$.
(3) The curvature K_g is bounded above by a negative constant.

Remark

The curvature K_g has lower bound -1 and $g_{\mathbb{D}}$ is bounded by g automatically.

- Benoist-Hulin 14' showed the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) for the hyperbolic affine sphere in $\mathbb{R}^3$.
- Labourie-Toulisse 20' showed the equivalence (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3) for the maximal surface in $\mathbb{H}^{2,n}$.
- Their proofs are in a geometric manner.
- Our proof uses the Higgs bundle techniques and Wan's PDE approach.

Sketch of Labourie-Toulisse's proof:

(a) K_g is bounded above by a negative constant.

(b) g is mutually bounded with $g_{\mathbb{D}}$.

(c) g is Gromov hyperbolic.

(d) There is no sequence $x_k \in X$, $g_k \in \mathrm{SO}_0(2, n+1)$ such that $g_k \cdot (X, x_k)$ Hausdorff converges to a flat surface (Barbot surface).

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(a)$\Rightarrow$(b): Ahlfors Lemma.
(b)$\Rightarrow$(c): Quasi-isometry preserves Gromov hyperbolicity.
(c)$\Rightarrow$(d): Gromov δ-hyperbolicity is a closed condition under the Hausdorff convergence.
(d)$\Rightarrow$(a): Compactness + Continuity + Rigidity. Involved.

Now we construct a Higgs bundle from a maximal surface in $\mathbb{H}^{2,n}$ (Collier-Tholozan-Toulisse).
First we recall the definition of Higgs bundle. Let Σ be a Riemann surface and K be the canonical line bundle of Σ.

Definition

A Higgs bundle over Σ is a pair (E, ϕ), where E is a holomorphic bundle over Σ and $\phi : E \to E \otimes K$ is holomorphic.

- Let H be a Hermitian metric on E.
- Hitchin equation: $F^{\nabla_H} + [\phi, \phi^{*H}] = 0$, where F^{∇_H} is the curvature of the Chern connection ∇_H with respect to H.
- Equivalently $D^2 = 0$ for $D = \nabla_H + \phi + \phi^{*H}$.
- ∇_H is a $U(n)$-connection and $\phi + \phi^{*H}$ is Hermitian w.r.t H.
- We call such H a harmonic metric.

- $\mathbb{R}^{2,n+1}$: $\mathbb{R}^{n+3}$ with quadratic form Q of signature $(2, n+1)$.
- $\mathbb{H}^{2,n}$: $\{x \in \mathbb{R}^{2,n+1} : Q(x) = -1\}$.
- An immersed surface S is spacelike if Q is positive on TS.
- Consider a spacelike conformal immersion of $\mathbb{D}$ with image Σ

$$f : \mathbb{D} \to \mathbb{H}^{2,n} \hookrightarrow \mathbb{R}^{2,n+1}.$$

- The pullback bundle can be decomposed as

$$\mathbb{D} \times \mathbb{R}^{2,n+1} = T \oplus N \oplus L.$$

- T: the tangent bundle of Σ.
- N: the normal bundle of Σ in $\mathbb{H}^{2,n}$.
- L: the position line bundle of Σ. $L = \mathcal{O}$.

- g_T, g_N, g_L: Riemannian metric on T, N, L from Q.
- The standard flat connection of $\mathbb{R}^{2,n+1}$ is decomposed as

$$D = \begin{pmatrix} \nabla_T & -\Pi^{*Q} & -B^{*Q} \\ \Pi & \nabla_N & 0 \\ B & 0 & \nabla_L \end{pmatrix}, \text{ second fundamental form } \Pi.$$

- Maximal surface: $\mathrm{tr}_{g_T}\Pi = 0$.
- $T^{\mathbb{C}}, N^{\mathbb{C}}, L^{\mathbb{C}}, g_T^{\mathbb{C}}, g_N^{\mathbb{C}}, g_L^{\mathbb{C}}, D^{\mathbb{C}}$: Complexification.
- h_T, h_N, h_L: Hermitian metrics from $g_T^{\mathbb{C}}, g_N^{\mathbb{C}}, g_L^{\mathbb{C}}$.
- $T^{\mathbb{C}} = \overline{K^{-1}} \oplus K^{-1} \cong K \oplus K^{-1}$, $h_T = (h_K, h_K^{-1})$.
- $H = (h_K, h_K^{-1}, h_N, h_L)$.
- $\nabla_T = \begin{pmatrix} \nabla_K & 0 \\ 0 & \nabla_{K^{-1}} \end{pmatrix}$, $\Pi = (\alpha^{*Q*H} \;\; \alpha)$, $B = (1 \;\; 1^{*H})$.
- $D^{\mathbb{C}} = \nabla_H + \Phi_H$ from $gl(n,\mathbb{C}) = u(n) \oplus herm$ w.r.t H.

- $E = T^{\mathbb{C}} \oplus N^{\mathbb{C}} \oplus L^{\mathbb{C}} = K \oplus K^{-1} \oplus N^{\mathbb{C}} \oplus \mathcal{O}$.
- $\bar{\partial}_E = \nabla_H^{(0,1)} = \mathrm{diag}(\bar{\partial}_K, \bar{\partial}_{K^{-1}}, \nabla_{N^{\mathbb{C}}}^{(0,1)}, \bar{\partial})$.
- From maximality, $\Pi(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$, so Π is $(2,0)+(0,2)$.
- Higgs field $\phi = \Phi_H^{(1,0)} = \begin{pmatrix} 0 & 0 & \beta^\dagger & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, where $\beta = \alpha$ is the $(2,0)$-part of Π.
- From $D^2 = 0$, $\beta \in \mathrm{Hom}(K^{-1}, N^{\mathbb{C}}) \otimes K$ is holomorphic.
- $(E, \bar{\partial}_E, \phi)$ is a Higgs bundle.
- $q_4 = g_N^{\mathbb{C}}(\beta, \beta)$ is a holomorphic quartic differential.

Theorem (D-Li 22')

Let $f : \mathbb{D} \to \mathbb{H}^{2,n}$ be a spacelike conformal maximal surface with the induced metric $g = g_T$. Let q_4 be the holomorphic quartic differential defined above. Suppose g is complete. Then the followings are equivalent.
(1) q_4 is bounded with respect to $g_{\mathbb{D}}$.
(2) g is bounded by $g_{\mathbb{D}}$.
(3) The curvature K_g is bounded above by a negative constant.

- $K_g = -1 + |\beta|_H^2 \geq -1$ from Gauss Equation.
- $g_{\mathbb{D}}$ is bounded by g from Ahlfors Lemma.

Proof: Let (E, ϕ) be a Higgs bundle with a harmonic metric H.

Lemma

For a holomorphic section s of $End(E)$, locally $\phi = \varphi dz$, we have

$$\partial_z \partial_{\bar{z}} \log |s|_H^2 \geq \frac{|[s, \varphi^{*_H}]|_H^2 - |[s, \varphi]|_H^2}{|s|_H^2}.$$

Letting $s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta(\frac{\partial}{\partial z}) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and using the formula of K_g, we obtain

$$\triangle_g K_g \geq K_g^2 + K_g.$$

Applying the Cheng-Yau maximum principle, we obtain $K_g \leq 0$.

Lemma (Cheng-Yau maximum principle, rough version)

Let (M, g) *be a complete Riemannian manifold with Ricci curvature bounded below. For a smooth function* u *satisfying* $\triangle_g u \geq F(u)$, *where* F *has growth rate greater than linear growth, then* u *has an upper bound and* $F(\sup u) \leq 0$.

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Then (2)⇒(3) follows from the following lemma.

Lemma (Key estimate)

Let g be a Riemannian metric on $\mathbb{D}$ such that $C^{-1}g \leq g_{\mathbb{D}} \leq Cg$. Suppose K_g satisfies $-a \leq K_g \leq 0$ and $\triangle_g K_g \geq cK_g$ for some constants $a, c > 0$, then there is a constant $\delta = \delta(C, a, c) > 0$ such that $K_g \leq -\delta$.

Proof of the key estimate: We use Wan's approach in the proof of the case of harmonic maps.

- g being equivalent to $g_{\mathbb{D}}$ implies the area of disk $D_g(x, r)$ has exponential growth.
- There exists $R > 0$, such that for every $x \in \mathbb{D}$, there exists $r \in (0, R)$ satisfying $\frac{d^2}{dr^2}\text{Area}(D_g(x, r)) \geq 2\pi + 1$.
- $\frac{d^2}{dr^2}\text{Area}(D_g(x, r)) = \frac{d}{dr}\text{L}(\partial D_g(x, r)) = \int_{\partial D_g(x,r)} k_g ds_g$.
- Gauss-Bonnet-Chern: $\int_{\partial D_g(x,r)} k_g ds_g = 2\pi - \int_{D_g(x,r)} K_g dA_g$.
- L^1 estimate of $-K_g$: $\int_{D_g(x,r)}(-K_g)dA_g \geq 1$.

- Mean-value inequality for $\triangle_g(-K_g) \leq c(-K_g)$:
  $\exists$ constants $0 < p < 1$, $C > 0$ such that for every $x \in \mathbb{D}$,

$$-K_g(x) \geq C\left(\int_{D_g(x,R)} (-K_g)^p dA_g\right)^{\frac{1}{p}}.$$

- Together with $-K_g \leq a$, L^p norm controls L^1 norm.
- $-K_g \geq \delta$ for some constant $\delta > 0$.

(3)⇒(2): Ahlfors Lemma.

(2)⇒(1): $C^{-1}|q_4|_{g_{\mathbb{D}}} \leq |q_4|_g = |g_N^{\mathbb{C}}(\beta,\beta)|_g = |\beta|_H^2 = 1 + K_g \leq 1.$

(1)⇒(2): Recall $\phi = \begin{pmatrix} 0 & 0 & \beta^\dagger & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

- $\mathrm{tr}(\phi^{4k+i}) = 0, i = 1,2,3$, and $\mathrm{tr}(\phi^{4k}) = 4q_4^k$. Bounded.
- Li-Mochizuki 20': Bounded $\mathrm{tr}\phi^j$ for all j implies bounded ϕ.
- $|\phi|_{H,g_{\mathbb{D}}} \leq C$. In particular, $|1|_{H,g_{\mathbb{D}}} \leq C$.
- $1 \in H^0(\mathbb{D}, K^{-1} \otimes K)$. $|1|_{H,g_{\mathbb{D}}} = |\frac{\partial}{\partial z}|_g |dz|_{g_{\mathbb{D}}} \leq C$.
- $g \leq C g_{\mathbb{D}}$.

Remark

By lifting to the universal cover, the theorem above also holds for general hyperbolic Riemannian surfaces. But if we consider the complex plane $\mathbb{C}$*, the story is totally different since there is no bounded holomorphic differentials on* $\mathbb{C}$*.*

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By lifting to the universal cover, the theorem above also holds for general hyperbolic Riemannian surfaces. But if we consider the complex plane $\mathbb{C}$, the story is totally different since there is no bounded holomorphic differentials on $\mathbb{C}$.

Remark

It seems that the conditions (2)(3) are not coincided with Wan's result. This is because in the case of harmonic maps, the induced metric is not conformal to $g_{\mathbb{D}}$, while in the cases of other geometries, the induced metric is conformal.

- (2) The energy density $|df|^2$ is bounded. (q_2)
  (2) The induced metric g is bounded by $g_{\mathbb{D}}$. (q_3, q_4, q_6)
- The energy density is just the $(1,1)$-part (conformal part) of the induced metric.
- (3) The harmonic map f is quasi-conformal. (q_2)
  (3) The curvature K_g is bounded above by a negative constant. (q_3, q_4, q_6)
- The harmonic map being quasi-conformal is equivalent to the curvature of $\hat{g} = |\partial f|^2 g_{\mathbb{D}}$ (complete) being bounded above by a negative constant.

- We come back to the harmonic map $f : \mathbb{D} \to \mathbb{D}$.
- Suppose $\partial f \neq 0$. Let $w = -\log|\partial f|$.
- Then $\triangle_{g_{\mathbb{D}}} w = |q|^2_{g_{\mathbb{D}}} e^{2w} - e^{-2w} + \frac{1}{4}$.
- It is a Toda equation.

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- It is a Toda equation.
- We generalize Wan's result to the Toda system.

$$
\begin{aligned}
\triangle_{g_{\mathbb{D}}} w_1 &= |q|^2_{g_{\mathbb{D}}} e^{w_1 - w_r} - e^{w_2 - w_1} + \frac{r-1}{4}, \\
\triangle_{g_{\mathbb{D}}} w_i &= e^{w_i - w_{i-1}} - e^{w_{i+1} - w_i} + \frac{r+1-2i}{4}, \ 2 \leq i \leq r-1 \\
\triangle_{g_{\mathbb{D}}} w_r &= e^{w_r - w_{r-1}} - |q|^2_{g_{\mathbb{D}}} e^{w_1 - w_r} + \frac{1-r}{4}.
\end{aligned}
$$

Theorem (Li-Mochizuki 20')

Let q be a holomorphic r-differential on $\mathbb{D}$. Then there is a unique complete solution $(w_1, \cdots, w_r)$ to the Toda system. For the complete solution, we mean that $g_i = e^{-w_i + w_{i+1}} g_{\mathbb{D}}$, $1 \leq 1 \leq r - 1$ are complete.

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The Toda system is just the Hitchin equation of the cyclic Higgs bundle in the Hitchin section.

$$E = K^{\frac{r-1}{2}} \oplus K^{\frac{r-3}{2}} \oplus \cdots \oplus K^{\frac{3-r}{2}} \oplus K^{\frac{1-r}{2}}, \quad \phi = \begin{pmatrix} & & & q \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \end{pmatrix}.$$

From the "non-Abelian Hodge correspondence", a solution to the Hitchin equation corresponds to a harmonic map

$$f : \mathbb{D} \to SL(r, \mathbb{C})/SU(r).$$

Denote g as the pullback metric of f.

Theorem

Consider the Toda system for $r \geq 3$ with the complete solution $(w_1, \cdots, w_r)$. Then the followings are equivalent:
(1) q is bounded with respect to $g_{\mathbb{D}}$;
(2) g is bounded by $g_{\mathbb{D}}$;
(3) The curvature K_g is bounded above by a negative constant.

Remark

The proof mainly uses the key lemma above and the estimate of Li-Mochizuki 20':

$$e^{-w_{i-1}+w_i} < e^{-w_i+w_{i+1}},\ 1 \le i \le [\frac{r}{2}].$$

Remark

We also obtain similar result to the case of subcyclic Higgs bundles, which contains the situation of q_6.

Remark

Let Σ be a hyperbolic Riemann surface. Let g be a complete Hermitian metric on Σ with curvature $-a \le K_g \le -b$ for some constants $a, b > 0$. Then the theorem above holds.

Thanks for your attention.